SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3276. [2007 : 428, 430] *Proposed by Neven Jurič, Zagreb, Croatia.*

A sequence $\{a_n\}_{n=0}^{\infty}$ of positive real numbers satisfies the recurrence relation $a_{n+3} = a_{n+1} + a_n$ for $n \geq 0$. Simplify

$$\sqrt{a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 - a_{n+2}^2 + a_{n+1}^2 - a_n^2}\,.$$

Solution submitted independently by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.

Clearly, all the terms of the given sequence are positive. Set

$$A \;=\; \sqrt{a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 - a_{n+2}^2 + a_{n+1}^2 - a_n^2}\,.$$

By the recurrence relation, $a_{n+3} = a_{n+1} + a_n$, $a_{n+4} = a_{n+2} + a_{n+1}$, and $a_{n+5} = a_{n+3} + a_{n+2} = a_{n+2} + a_{n+1} + a_n$, so that

$$\begin{aligned} & a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 \\ &\quad= (a_{n+2} + a_{n+1} + a_n)^2 + (a_{n+2} + a_{n+1})^2 + (a_{n+1} + a_n)^2 \\ &\quad= 2a_{n+2}^2 + 3a_{n+1}^2 + 2a_n^2 + 4a_{n+2}a_{n+1} + 2a_{n+2}a_n + 4a_{n+1}a_n\,. \end{aligned}$$

Thus,

$$\begin{aligned} A^2 &= a_{n+2}^2 + 4a_{n+1}^2 + a_n^2 + 4a_{n+2}a_{n+1} + 2a_{n+2}a_n + 4a_{n+1}a_n \\ &= (a_{n+2} + 2a_{n+1} + a_n)^2\,. \end{aligned}$$

Therefore,

$$\begin{aligned} A &= a_{n+2} + 2a_{n+1} + a_n \\ &= (a_{n+2} + a_{n+1}) + (a_{n+1} + a_n) \\ &= a_{n+4} + a_{n+3} \\ &= a_{n+6}\,. \end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, KARL HAVLAK and PAULA KOCA, Angelo State University, San Angelo, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS,